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Quartic spline-on-spline interpolation[1]

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Abstract

This paper provides a class of quartic spline-on-spline interpolation $S^{[k]}(x)$ on the function $f(x)$, $k = 0(1)4$. Under the end conditions selected appropriately, we show that when the knots are equally spaced, with spacing h, then the order of accuracy of the spline-on-spline $S^{[0]}(x)$, $S^{[1]}(x)$ approximations $f(x), f'(x)$ are $O(h^5)$, respectively, and the order of accuracy of the spline-on-spline $S^{[k]}(x)$ approximations $f^{(k)}(x)$ are $O(h^4)$, $k = 2,3,4$.

Keywords: Quartic spline; Spline-on-spline; Interpolation

AMS classification: 41A60, 65L10

1. Introduction

In the recent spline analysis study, everybody pays attention to developing the spline-on-spline technique for approximating the first (or higher order) derivative of a function, because there is computational evidence that this yields better results than the traditional process does by using a single spline [1]. For odd degree spline, Dolezal and Tewarson [3] have obtained error bounds for spline-on-spline interpolation. On a uniform mesh, with spacing h, Papamichael and Soares [4] and Chen [2] have obtained the order of error approximating derivatives of a function with their cubic spline-on-spline interpolation. Sakai [5] derived asymptotic expansions of the error for the quintic spline-on-spline interpolation. Under appropriate end conditions we obtained error uniform estimates of the quintic spline-on-spline interpolation [6]. The object of this paper is to derive error uniform estimates of a class of the quartic spline-on-spline. Let Δ denote the uniform partition Δ: $x_i = a + ih$, $i = 0(1)n$, $h = (b - a)/n$, and let $\mathrm{Sp}(\Delta, 4)$ denote the space of all quartic splines on $[a,b]$ with partition Δ. We discuss the interpolating problem of the quartic spline: finding

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$S \in \mathrm{Sp}(\Delta, 4)$ satisfying the interpolation conditions (1) $S(x_{i+1/2}) = f_{i+1/2}$, $i = 0(1)n-1$, (2) $D^j S(x_0) = S_{0,j}$, $D^j S(x_n) = S_{n,j}$, $j = 0, 2$, where $D^j = \mathrm{d}^j/\mathrm{d}x^j$, $x_{i+1/2} = x_i + \frac{1}{2}h$. For simplicity denote the above interpolation spline as $S(x) = \mathrm{Sp}(f_{i+1/2}, i = 0(1)n-1, S_{0,j}, S_{n,j}, j = 0, 2)$.

In order to define quartic spline-on-spline, we must introduce the following Lagrange interpolation polynomial on the interval $[0, 1]$: finding $L \in \wp_5$, such that $L(j/6) = g(j/6)$, $j = 0, 1, 2, 4, 5, 6$, where $\wp_k$ denotes the space of all the polynomials of degree at most k. Define

$$S^{[1]}(x) = \mathrm{Sp}(DL_{i+1/2}, i = 0(1)n-1, S^{[1]}_{0,j}, S^{[1]}_{n,j}, j = 0, 2),$$

$$S^{[k]}(x) = \mathrm{Sp}(D^2 S^{[k-2]}_{i+1/2}, i = 0(1)n-1, S^{[k]}_{0,j}, S^{[k]}_{n,j}, j = 0, 2),$$

where $D^\alpha g(x_i) = D^\alpha g_i$, and $L(x)$ denotes the piecewise Lagrange interpolation polynomial of $S(x)$ on partition Δ. We say that $S^{[k]}$ are k-fold quasi-quartic spline-on-spline interpolations of $f(x)$, $k = 0(1)4$. Under some special end conditions, using superconvergence nature of the spline interpolation at the semi-knots $x_{i+1/2}$, we obtained that the order of accuracy of the spline-on-spline $S^{[0]}(x)$, $S^{[1]}(x)$ approximations $f(x), f'(x)$ are $O(h^5)$, respectively, and the order of $S^{[k]}(x)$ approximations $f^{(k)}(x)$, $k = 2, 3, 4$, are $O(h^4)$ (cf. Theorems 6 and 7).

2. Some lemma

Let us introduce the $(n+1)$ order matrix

$$A = \begin{pmatrix} 115 & 76 & 1 & & & & \\ 306 & 383 & 78 & 1 & & & \\ 1 & 76 & 230 & 76 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 76 & 230 & 76 & 1 \\ & & & 1 & 78 & 383 & 306 \\ & & & & 1 & 76 & 115 \end{pmatrix}.$$

Lemma 1. *The $(n+1)$ order matrix is nonsingular, and the uniform norm of its inverse has an estimate*

$$\| A^{-1} \|_\infty \leqslant C, \tag{2.1}$$

where C is a constant independent of n.

Proof. Let $P(x) = x^2 + 76x + 115$, $Q(x) = x^3 + 78x^2 + 383x + 306$. It is easy to see that the quartic polynomial $x^4 + 76x^3 + 230x^2 + 76x + 1$ has four zeros

$$\lambda_1 = \lambda_4^{-1} = \tfrac{1}{2}(b + \sqrt{b^2 - 4}) \approx -0.0137, \quad \lambda_2 = \lambda_3^{-1} = \tfrac{1}{2}(a + \sqrt{a^2 - 4}) \approx -0.3615,$$

where $a = -38 + 8\sqrt{19}$, $b = -38 - 8\sqrt{19}$. By [5], in order to prove Lemma 1, we only need to verify

$$K = P(\lambda_1)\cdot Q(\lambda_2) - P(\lambda_2)\cdot Q(\lambda_1) \neq 0.$$

In fact, by the above definition of $P(x)$ and $Q(x)$, and λ_1, λ_2, we can directly verify $K < 0$. The proof is complete. □

Define the following sequences:

$$\alpha_{2k+1} = -\tfrac{1}{192}\left[144\,\frac{9^{k+1}-1}{4^k(2k+3)!} + \sum_{j=0}^{k-1}\frac{76+4^{k-j}}{(2k-2j)!}\alpha_{2j+1}\right], \quad k \geqslant 0,$$

$$\beta_{k,0} = 144\,\frac{3^{k+2}-1}{2^k(k+3)!} + \sum_{j=0}^{[(k-1)/2]}\frac{76+2^{k-2j}}{(k-2j)!}\alpha_{2j+1} + 192\alpha_{k+1}\cdot\delta_{k,2[k/2]}, \quad k \geqslant 2, \tag{2.2}$$

$$\beta_{k,1} = 48\,\frac{3^{k+3}-1+(-1)^k(2-2^{k+4})}{2^k(k+3)!} + \frac{672\cdot(-1)^k}{(k+1)!} + 768\alpha_{k+1}\cdot\delta_{k,2[k/2]}$$
$$+ \sum_{j=0}^{[(k-1)/2]}\frac{78+2^{k-2j}+(-1)^k\cdot 306}{(k-2j)!}\alpha_{2j+1}, \quad k \geqslant 2.$$

Here $[x]$ denotes the largest integer which does not exceed x, and $\delta_{i,j}$ is the Kronecker function,

$$\delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Particularly, we have $\alpha_1 = -1$, $\alpha_3 = \frac{1}{12}$, $\alpha_5 = -\frac{1}{240}$, $\beta_{2,0} = \beta_{2,1} = 0$. (2.3)

Introduce functional

$$d_0(g) = \frac{384}{h^3}\left[g_{3/2} - 3g_{1/2} + 2g_0 - \tfrac{3}{4}h^2D^2g_0\right],$$

$$d_1(g) = \frac{384}{h^3}\left[2g_0 - 2g_{1/2} - g_{3/2} + g_{5/2} - \tfrac{7}{4}h^2D^2g_0\right],$$

$$d_i(g) = \frac{384}{h^3}\left[g_{i+3/2} - 3g_{i+1/2} + 3g_{i-1/2} - g_{i-3/2}\right], \quad i = 2(1)n-2, \tag{2.4}$$

$$d_{n-1}(g) = \frac{384}{h^3}\left[2g_n - 2g_{n-1/2} - g_{n-3/2} + g_{n-5/2} - \tfrac{7}{4}h^2D^2g_n\right],$$

$$d_n(g) = \frac{384}{h^3}\left[g_{n-3/2} - 3g_{n-1/2} + 2g_n - \tfrac{3}{4}h^2D^2g_n\right],$$

and

$$\tau_i(g,m) = d_i(g) + \sum_{k=0}^{m} \alpha_{2k+1} h^{2k} [D^{2k+3} g_{i-2} + 76 D^{2k+3} g_{i-1} + 230 D^{2k+3} g_i + 76 D^{2k+3} g_{i+1} + D^{2k+3} g_{i+2}], \quad i = 2(1)n-2,$$

$$\tau_0(g,m) = d_0(g) + \sum_{k=0}^{m} \alpha_{2k+1} h^{2k} [115 D^{2k+3} g_0 + 76 D^{2k+3} g_1 + D^{2k+3} g_2],$$

$$\tau_1(g,m) = d_1(g) + \sum_{k=0}^{m} \alpha_{2k+1} h^{2k} [306 D^{2k+3} g_0 + 383 D^{2k+3} g_1 + 78 D^{2k+3} g_2 + D^{2k+3} g_3], \tag{2.5}$$

$$\tau_{n-1}(g,m) = d_{m-1}(g) + \sum_{k=0}^{m} \alpha_{2k+1} h^{2k} [306 D^{2k+3} g_n + 383 D^{2k+3} g_{n-1} + 78 D^{2k+3} g_{n-2} + D^{2k+3} g_{n-3}],$$

$$\tau_n(g,m) = d_n(g) + \sum_{k=0}^{m} \alpha_{2k+1} h^{2k} [115 D^{2k+3} g_n + 76 D^{2k+3} g_{n-1} + D^{2k+3} g_{n-2}].$$

By Taylor series expansion, we can prove

Lemma 2. *Let $f \in C^{L+3}[a,b]$, $L = 2m+1$ or $2m+2$. Then*

$$\tau_i(g,m) = O(h^L), \quad i = 2(1)n-2.$$

$$\tau_j(g,m) = \sum_{k=3}^{L-1} \beta_{k,j} h^k D^{k+3} g(x_j) + O(h^L),$$

$$\tau_{n-j}(g,m) = \sum_{k=3}^{L-1} (-1)^k \beta_{k,j} h^k D^{k+3} g(x_{n-j}) + O(h^L), \quad j = 0, 1.$$

Let us denote $T(x,g) = D^3 g(x) - \frac{1}{2} h^2 D^5 g(x)$,

$$H_i(g) = \frac{1}{h}(g_{i+1/2} - g_{i-1/3}) - \frac{h^2}{384} [T(x_{i+1},g) + 14 T(x_i,g) + T(x_{i-1},g)]. \tag{2.6}$$

Using Taylor series expansion at the knots x_i, we get

Lemma 3. *Let $g \in C^{L+1}$, $L = 2v+1$ or $L = 2v+2$.* Then

$$H_i(g) = Dg(x_i) + \sum_{k=2}^{v} B_{2k+1} h^{2k} D^{2k+1} g(x_i) + O(h^L), \tag{2.7}$$

where

$$B_{2k+1} = \frac{1}{4^k (2k+1)!} + \frac{1}{192}\left[\frac{\alpha_1}{(2k-2)!} + \frac{\alpha_3}{(2k-4)!}\right] + \frac{1}{1152}\delta_{2,k}, \quad k \geqslant 2.$$

(Particularly, $B_5 = \frac{1}{6!}$.) On the interval $[0,1]$, *we consider the following Birkhoff interpolation problem: finding* $B \in \wp_4$, *such that*

$$B(\tfrac{1}{2}) = g(\tfrac{1}{2}),$$
$$D^j B(i) = D^j g(i), \quad i = 0,1,\ j = 1,3. \tag{2.8}$$

We say that the $B(t)$ *is a Birkhoff interpolation polynomial of the function* $g(x)$.

Lemma 4. *The solution of the problem* (2.8) *uniquely exists.*

Proof. It is sufficient to prove that the corresponding homogeneous problem has zero solution. Let $B \in \wp_4$, such that $B(\frac{1}{2}) = 0$, $D^j B(i) = 0$, $i = 0,1$, $j = 1,3$. By $B(\frac{1}{2}) = 0$, we know $B(t) = (2t-1)(at^3 + bt^2 + ct + d)$. Since $B^j(i) = 0$, $i = 0,1, j = 1,3$, the coefficients a, b, c, d must satisfy the following equations: $2d - c = 0$, $5a + 4b + 3c + 2d = 0$, $a = 2b$, $7a + 2b = 0$. Hence $a = b = c = d = 0$, we have $B(t) = 0$. This completes the proof of this lemma. □

It is easy to see that the Birkhoff interpolation polynomial $B(t)$ can be expressed as the following form:

$$B(t) = \sum_{i=0}^{1} [\varphi_{1,1}(t) Dg(i) + \varphi_{1,3}(t) D^3 g(i)] + \varphi_{1/2}(t) g(\tfrac{1}{2}), \tag{2.9}$$

where the bounded functions $\varphi_{1/2}(t)$ and $\varphi_{i,j}(t) \in \wp_4$, $i = 0,1, j = 1,3$, satisfy the conditions

$$\varphi_{1/2}(\tfrac{1}{2}) = 1,\ D^j \varphi_{1/2}(i) = 0, \quad \varphi_{i,j}(\tfrac{1}{2}) = 0, \qquad i = 0,1;\ j = 1,3,$$
$$D^k \varphi_{i,j}(l) = \delta_{i,l} \cdot \delta_{j,k}, \quad i,l = 0,1;\ j,k = 1,3.$$

Similarly, the Lagrange interpolation polynomial $L(t)$ can be expressed as the following form:

$$L(t) = \sum_{j=0,\, j \neq 3}^{6} l_{j/6}(t) \cdot g(j/6), \tag{2.10}$$

where $l_{j/6}(t) \in \wp_5$, $j = 0(1)6$, $j \neq 3$, satisfy the conditions $l_{j/6}(j/6) = \delta_{i,j}$, $i,j = 0(1)6$, $i,j \neq 3$ (cf. Section 1).

3. Multiply quartic spline-on-spline

Let $f \in C^8[a,b]$ and denote $M_i^{[k]} = D^3 S^{[k]}(x_i)$, $i = 0(1)n$, where $S^{[k]}(x)$ is the k-fold quartic spline-on-spline interpolating (cf. Section 1). By [1], we have the following work equations:

$$A \cdot M^{[k]} = D^{[k]}, \tag{3.1}$$

where the $(n+1)$ order matrix A is defined by (2.1), $M^{[k]}$ and $D^{[k]}$ are the $(n+1)$ dimensional vectors as follows, respectively:

$$M^{[k]} = (M_0^{[k]}, M_1^{[k]}, \ldots, M_n^{[k]})^{\mathrm{T}}; \quad D^{[k]} = (d_0^{[k]}, d_1^{[k]}, \ldots, d_n^{[k]})^{\mathrm{T}}.$$

Here $d_i^{[k]} = d_i(S^{[k]})$, $i = 0(1)n$, and functional $d_i(g)$ are defined by (2.4).

We can derive the following relation expressions of the quartic spline interpolation function at the knots x_i:

$$S^{[k]}(x_i) = \frac{1}{8}\left[\frac{h^3}{384}(3M_{i-1}^{[k]} + 115M_i^{[k]} + 73M_{i+1}^{[k]} + M_{i+1}^{[k]}) + 6S_{i+1/2}^{[k]} - S_{i+3/2}^{[k]} + 3S_{i-1/2}^{[k]}\right], \tag{3.2}$$

$$DS_i^{[k]} = \frac{1}{h}(S_{i+1/2}^{[k]} - S_{i-1/2}^{[k]}) - \frac{h^2}{384}(M_{i-1}^{[k]} + 14M_i^{[k]} + M_{i+1}^{[k]}), \tag{3.3}$$

$$D^2S_{i+1/2}^{[k]} = \frac{1}{h^2}(S_{i+3/2}^{[k]} - 2S_{i+1/2}^{[k]} + S_{i-1/2}^{[k]}) + \frac{h}{384}(M_{i-1}^{[k]} + 29M_i^{[k]} - 29M_{i+1}^{[k]} - M_{i+2}^{[k]}). \tag{3.4}$$

Let the interpolation spline function $S^{[0]}(x)$ satisfy the following end conditions ($j = 0, n$):

$$\begin{aligned} S^{[0]}(x_j) &= f(x_j) + (\delta_{0,j} - \delta_{n,j})\cdot a_0h^6D^6f(x_j) + b_0h^7D^7f(x_j),\\ D^2S^{[0]}(x_j) &= D^2f(x_j) + (\delta_{0,j} - \delta_{n,j})\cdot a_1h^4D^6f(x_j) + b_1h^5D^7f(x_j). \end{aligned} \tag{3.5}$$

Set

$$\begin{aligned}
\tau_0^{[0]} &= \tau_0(f,2) - 288(a_1h^3D^6f_0 + b_1h^4D^7f_0) + 768(a_0h^3D^6f_0 + b_0h^4D^7f_0),\\
\tau_1^{[0]} &= \tau_1(f,2) - 672(a_1h^3D^6f_0 + b_1h^4D^7f_0) + 768(a_0h^3D^6f_0 + b_0h^4D^7f_0),\\
\tau_i^{[0]} &= \tau_i(f,2), \quad i = 2(1)n-2,\\
\tau_{n-1}^{[0]} &= \tau_{n-1}(f,2) + 672(a_1h^3D^6f_n - b_1h^4D^7f_n) - 768(a_0h^3D^6f_n - b_0h^4D^7f_0),\\
\tau_n^{[0]} &= \tau_n(f,2) + 288(a_1h^3D^6f_n - b_1h^4D^7f_n) - 768(a_0h^3D^6f_n - b_0h^4D^7f_n).
\end{aligned}$$

By Lemma 2, we easily obtain

Lemma 5. *Let $f \in C^8[a,b]$. If*

$$a_0 = \frac{3\beta_{31} - 7\beta_{30}}{3072}, \quad b_0 = \frac{3(\beta_{31} + \beta_{41}) - 7\beta_{40}}{3072}, \quad a_1 = \frac{\beta_{31} - \beta_{30}}{384}, \quad b_1 = \frac{\beta_{31} + \beta_{41} - \beta_{40}}{384}, \tag{3.6}$$

then $\tau_i^{[0]} = O(h^5)$, $i = 0(1)n$.

Substituting the end conditions of the spline $S^{[0]}$ into the work equation (3.1) (when $k = 0$), and by Lemma 5, we have

$$A \cdot R^{[0]} = T^{[0]},$$

where $T^{[0]} = (\tau_0^{[0]}, \tau_1^{[0]}, \ldots, \tau_n^{[0]})^{\mathrm{T}} = O(h^5)(1, 1, \ldots, 1)^{\mathrm{T}}$, $R^{[0]} = (R_0^{[0]}, R_1^{[0]}, \ldots, R_n^{[0]})^{\mathrm{T}}$, here $R_i^{[0]} = M_i^{[0]} - D^3f(x_i) + \alpha_3h^2D^5f_i + \alpha_5h^4D^7f_i$, $i = 0(1)n$. Therefore, by Lemma 1 and (2.3), when

$f \in C^8[a,b]$, we have

$$D^3S^{[0]}(x_i) - D^3f(x_i) = -\tfrac{1}{12}h^2D^5f(x_i) - \alpha_5 h^4 D^7 f(x_i) + \mathrm{O}(h^5). \tag{3.7}$$

Substituting (3.7) into (3.3) (when $k = 0$), and using Taylor series expansion at the knots x_i, we have

$$S^{[0]}(x_i) = f(x_i) + \tfrac{1}{3072}h^6D^6f(x_i) + \mathrm{O}(h^7). \tag{3.8}$$

From (3.3) (when $k = 0$), (3.7), we get $DS^{[0]}(x_i) = H_i(f) + \mathrm{O}(h^6)$, where $H_i(g)$ are defined by (2.6). Thus by Lemma 3, we have

$$DS^{[0]}(x_i) - Df(x_i) = \mathrm{O}(h^4). \tag{3.9}$$

Substituting (3.7) into (3.4) (when $k = 0$), and using Taylor series expansion at the semi-knots $x_{i+1/2}$, we have

$$D^2S^{[0]}(x_{i+1/2}) - D^2f(x_{i+1/2}) = \tfrac{7}{1920}h^4D^6f(x_{i+1/2}) + \mathrm{O}(h^6). \tag{3.10}$$

By the above results, we have

Theorem 6. *Let $f \in C^8[a,b]$, then*

$$\| f - S^{[0]} \| = \mathrm{O}(h^5), \tag{3.11}$$

and we have the following asymptotic asymptotic expansion at the semi-knots $x_{i+1/2}$:

$$D^2S^{[0]}(x_{i+1/2}) - D^2f(x_{i+1/2}) = \tfrac{7}{1920}h^4D^6f(x_{i+1/2}) + \mathrm{O}(h^6). \tag{3.12}$$

Proof. We only prove (3.11). For any subinterval $[x_i, x_{i+1}]$, we consider the following Birkhoff interpolation: finding $B(f,x) \in \wp_4$, such that

$$B(f, x_{i+1/2}) = f(x_{i+1/2}), \quad D^jB(f,x_k) = D^jf(x_k), \quad j = 1,3; \; k = i, i+1. \tag{3.13}$$

By Lemma 4, we know that the interpolation problem (3.13) uniquely exists. It is obvious that

$$f(x) - B(f,x) = \mathrm{O}(h^5), \quad x \in [x_i, x_{i+1}], \tag{3.14}$$

and $B(S^{[0]}, x) \equiv S^{[0]}(x)$. By (2.9), we have

$$\begin{aligned} B(f,x) - S^{[0]}(x) &= B(f - S^{[0]}, x) \\ &= \sum_{j=0}^{1} [h\varphi_{j,1}(Df(x_{i+j}) - DS^{[0]}(x_{i+j})) + h^3\varphi_{j,3}(t)(D^3f(x_{i+j}) - D^3S^{[0]}(x_{i+j}))] \\ &\quad + \varphi_{1/2}(t)(f(x_{i+1/2}) - S^{[0]}(x_{i+1/2})), \end{aligned}$$

where $t = (x - x_i)/h$. By (3.7), (3.9), and the interpolation conditions of $S^{[0]}(x)$ we have proved (3.11). The proof is complete. □

Next, let us suppose that the k-fold quartic spline-on-spline $S^{[0]}(x)$, $k = 1,2,3,4$, satisfies the end conditions ($j = 0, n$):

(1) $k = 1$, $S^{[1]}(x_j) = Df(x_j)$, $D^2 S^{[1]}(x_j) = D^3 f(x_j)$.

(2) $k = 2$, $S^{[2]}(x_j) = D^2 f(x_j) + \frac{7}{1920} h^4 D^6 f(x_j)$, $D^2 S^{[2]}(x_j) = D^4 f(x_j)$.

(3) $k = 3$, $S^{[3]}(x_j) = D^3 f(x_j)$, $D^2 S^{[3]}(x_j) = D^5 f(x_j)$.

(4) $k = 4$, $S^{[4]}(x_j) = D^4 f(x_j)$, $D^2 S^{[4]}(x_j) = D^6 f(x_j)$.

Theorem 7. *Let $f \in C^8[a,b]$ and suppose that the quartic spline-on-spline $S^{[k]}(x)$, $k = 1,2$, satisfies the end conditions, respectively. Then we have:*

$$\begin{aligned}
k = 1:\ & \|Df - S^{[1]}\|_\infty = O(h^5),\\
& D^2 S^{[1]}(x_{i+1/2}) - D^3 f(x_{i+1/2}) = O(h^4), \quad i = 0(1)n-1.\\
k = 2:\ & \|D^2 f - S^{[2]}\|_\infty = O(h^4),\\
& D^2 S^{[2]}(x_{i+1/2}) - D^4 f(x_{i+1/2}) = O(h^4), \quad i = 0(1)n-1.\\
k = 3:\ & \|D^3 f - S^{[3]}\|_\infty = O(h^4).\\
k = 4:\ & \|D^4 f - S^{[4]}\|_\infty = O(h^4).
\end{aligned}$$

Proof. We only prove the $k = 1$ case. For $k = 2,3,4$, the proof is similar. Let us prolong smoothly $f(x)$ from $[a,b]$ to $[a-h, b+h]$ such that $f \in C^8[a-h, b+h]$, and define

$$\begin{aligned}
S_{-1}^{[0]} &= S^{[0]}(x_{-1}) = f(x_{-1}) + \tfrac{1}{3072} h^6 D^6 f(x_{-1}),\\
S_{-1/2}^{[0]} &= S^{[0]}(x_{-1/2}) = f(x_{-1/2}),\\
S_0^{[0]} &= S^{[0]}(x_0) = f(x_0) + \tfrac{1}{3072} h^6 D^6 f(x_0),
\end{aligned} \tag{3.15}$$

where $x_{-1} = a - h$, $x_{-1/2} = a - \frac{1}{2}h$. Similarly we define $S_n^{[0]}, S_{n+1/2}^{[0]}, S_{n+1}^{[0]}$.

Using (2.10), we well know that on the interval $[x_{i-1}, x_{i+2}]$, $i = 0(1)n-1$, the quintic Lagrange interpolation polynomial is denoted by

$$L(x) = \sum_{j=0, j\neq 3}^{6} l_{j/6}(t) \cdot S^{[0]}(x_{i-1} + \tfrac{1}{2} jh), \tag{3.16}$$

where $t = (x - x_{i-1})/3h$. By (3.10) and the above result, a simple calculation shows that

$$DL(x_{i+1/2}) = \frac{1}{30h}[S_{i+2}^{[0]} - S_{i-1}^{[0]} + 45(S_{i+1}^{[0]} - S_i^{[0]}) + 9(S_{i-1/2}^{[0]} - S_{i+3/2}^{[0]})]. \tag{3.17}$$

Using (3.8), noting $S_{\tau+1/2}^{[0]} = f_{\tau+1/2}$, $\tau = i-1,\ i+1$, and by Taylor series expansion at the semi-knots $x_{i+1/2}$, we have

$$DL(x_{i+1/2}) = Df(x_{i+1/2}) + O(h^6), \quad i = 0(1)n-1. \tag{3.18}$$

Substituting the end conditions of the spline $S^{[1]}(x)$ into the work equation (3.1) (when $k = 1$), we obtain

$$A \cdot R^{[1]} = T^{[1]},$$

where $T^{[1]} = (\tau_0^{[1]}, \tau_1^{[1]}, \dots, \tau_n^{[1]})^{\mathrm{T}}$, $R^{[1]} = (R_0^{[1]}, R_1^{[1]}, \dots, R_n^{[1]})^{\mathrm{T}}$, here $\tau_i^{[1]} = \tau_i(Df, 1) + \mathrm{O}(h^3) = \mathrm{O}(h^3)$. $R_i^{[1]} = M_i^{[1]} - D^4 f_i + \frac{1}{12} h^2 D^6 f_i$, $i = 0(1)n$.

By Lemma 1, when $f \in C^8[a, b]$, we have

$$D^3 S^{[1]}(x_i) - D^4 f(x_i) = -\tfrac{1}{12} h^2 D^6 f(x_i) + \mathrm{O}(h^3). \tag{3.19}$$

Substituting (3.18), (3.19) into (3.3) (when $k = 1$), and using Taylor series expansion at the knots x_i, we have $DS^{[1]}(x_i) = H_i(Df) + \mathrm{O}(h^4)$.

By Lemma 3, we obtain

$$DS^{[1]}(x_i) - D^2 f(x_i) = \mathrm{O}(h^4). \tag{3.20}$$

Substituting (3.18), (3.19) into (3.4) (when $k = 1$), and using Taylor seires expansion at the semi-knots $x_{i+1/2}$, we have

$$D^2 S^{[1]}(x_{i+1/2}) - D^3 f(x_{i+1/2}) = \mathrm{O}(h^3). \tag{3.21}$$

Next we make uniform estimates of the spline $S^{[1]}(x)$, similarly as in the proof of Theorem 6, we introduce the Birkhoff interpolation polynomial of the function $f'(x)$ on the interval $[x_i, x_{i+1}]$: finding $B(Df, x) \in \wp_4$, such that

$$B(Df, x_{i+1/2}) = Df(x_{i+1/2}), \quad D^j B(Df, x_k) = D^{j+1} f(x_{k+1/2}), \; j = 1, 3; \; k = i, i+1.$$

From Lemma 4, we know that the interpolation problem uniquely exists, and

$$Df(x) - B(Df, x) = \mathrm{O}(h^5), \quad x \in [x_i, x_{i+1}]. \tag{3.22}$$

It is obvious that $B(S^{[1]}, x) = S^{[1]}(x)$, $x \in [x_i, x_{i+1}]$. By (2.9), we have

$$\begin{aligned} B(Df, x) - S^{[1]}(x) &= B(Df - S^{[1]}, x) \\ &= \sum_{j=0}^{1} [h\varphi_{j,1}(t)(D^2 f(x_{i+j}) - DS^{[1]}(x_{i+j})) + h^3 \varphi_{j,3}(t)(D^4 f(x_{i+j}) - D^3 S^{[1]}(x_{i+j}))] \\ &\quad + \varphi_{1/2}(t)(Df(x_{i+1/2}) - S^{[1]}(x_{i+1/2})), \end{aligned}$$

where $t = (x - x_i)/h$. Noting $S^{[1]}(x_{i+1/2}) = DL(x_{i+1/2})$, and by (3.18), (3.19), (3.20), we have

$$B(Df, x) - S^{[1]}(x) = \mathrm{O}(h^5). \tag{3.23}$$

From (3.22), (3.23), we proved

$$\| Df - S^{[1]} \|_\infty = \mathrm{O}(h^5).$$

This completes the proof of Theorem 7. □

4. Remarks

1. In the practice computation, to avoid the difficulty of giving high order derivative values in the end conditions, we can use their corresponding approximate, such as the high order difference values instead. In this way, it only requires that the error order is between $D^k f_j, j = 0, n$ and their approximate values as $O(h^{8-k})$ $(k \leqslant 7)$. All of the conclusions in this paper still hold in the case.
2. In this paper the odd-fold spline-on-spline interpolations are completely independent and we can use the parallel computer method.
3. Using the methods of this paper, we can construct the quartic spline-on-spline to approximating any order derivative of the function $f(x)$ to $O(h^4)$. But numerical examples show that it is not suitable to use the quartic spline-on-spline for approximating high order derivatives, because of the effect of their rounding errors.

5. Numerical example

In this section, we verify our results on numerical example, with $f(x) = \exp(5x)$. Here, we only give the results of even fold spline-on-spline interpolation functions; the case of odd fold is analogous and more obvious from the theoretical point of view. The programs are coded in TURBO C+ +. The tests are done on an IBM PC.

The computational results are summarized in Table 1, with $[a,b] = [0,1]$ and $h = \frac{1}{80}$. We introduce relative errors as the following:

$$e2(x) = (D^2 f(x) - D^2 S(x))/D^2 f(x); \quad ee2(x) = (D^2 f(x) - S^{[2]}(x))/D^2 f(x);$$

$$ee4(x) = (D^4 f(x) - S^{[4]}(x))/D^4 f(x).$$

Table 1

i	$e2(x_{i-1/2})$	$e2(x_{i-1/4})$	$ee2(x_{i-1/4})$	$ee4(x_{i-1/2})$
0	− 5.561771e − 08	1.902697e − 06	6.931440e − 08	− 9.353255e − 07
10	− 5.562143e − 08	1.902698e − 06	− 5.580549e − 08	− 1.074246e − 07
20	− 5.562157e − 08	1.902699e − 06	− 5.582468e − 08	− 1.111826e − 07
30	− 5.562133e − 08	1.902699e − 06	− 5.582446e − 08	− 1.114111e − 07
40	− 5.562148e − 08	1.902699e − 06	− 5.582460e − 08	− 1.112483e − 07
50	− 5.562152e − 08	1.902699e − 06	− 5.582465e − 08	− 1.111790e − 07
60	− 5.562186e − 08	1.902698e − 06	− 5.582494e − 08	− 1.110224e − 07
70	− 5.561796e − 08	1.902703e − 06	− 5.580500e − 08	− 1.628350e − 07
79	− 4.919629e − 08	1.911781e − 06	− 6.065966e − 08	6.151136e − 07

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